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# Approximating the Best Approximation Operator

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## INTRODUCTION

Let $X$ be a compact Hausdorff space and let $C(X)$ denote the set of real-valued functions which are continuous on $X$. Let $C(X)$ be normed in the usual fashion, $\|f\| = \sup_{x\in X} |f(x)|$.
If $P$ is a closed linear subspace of $C(X)$, we set

$$d(f, P) = \inf_{p\in P} \| p - f \|. \tag{1}$$

Let $Tf = \{ p \in P \mid d(f, P) = \| f - p \|\}$. If each $f$ in $C(X)$ has a best uniform approximation in $P$, then the best approximation operator $T$ is well-defined on $C(X)$.

In many cases, best approximations need not be unique; so the operator $T$ is a set-valued rather than a point-valued function. Thus, it is natural to ask if the operator $T$ can be approximated in some useful sense by a point-valued function. In what follows, we give some conditions which guarantee that $T$ has a best approximation by a continuous point-valued function.

If $\varphi$ is a point-valued function, $\varphi : C(X) \to P$ and $F$ a set-valued function, $F : C(X) \to 2^P$, then we define

$$\sigma(\varphi(f), F(f)) = \sup_{p\in F(f)} \| \varphi(f) - p \|, \tag{2}$$

$$\rho'(\varphi, F) = \sup_{f\in C(X)} \sigma(\varphi(f), F(f)). \tag{3}$$

The definition of $\sigma$ is the natural one with respect to the supremum norm on $C(X)$. We call $\varphi_0$ a best approximation to $F$ if $\rho'(\varphi_0, F) \leqslant \rho'(\varphi, F)$ for all point-valued functions $\varphi$.

Let $\lambda$ be a real positive scalar. If $f$ has more than one best approximation then clearly the diameter of the set $T\lambda f$ is $\lambda$ times the diameter of $Tf$ since $\lambda p \in T\lambda f$ whenever $p \in Tf$. Therefore, $\rho'(\varphi, T) = \infty$ for all point functions $\varphi$, so in order to obtain a sensible solution to the problem of approximating $T$ we must restrict the size of $Tf$. To this end, let

$$C(X, R) = \{f \in C(X) \mid \|f\| < R, R > 0\}.$$

We now define

$$\rho(\varphi, F) = \sup_{f \in C(X,R)} \sigma(\varphi(f), F(f)) \tag{4}$$

and say $\varphi_0$ is a best approximation to $F$ if $\rho(\varphi_0, F) \leqslant \rho(\varphi, F)$ for all functions $\varphi$ mapping $C(X, R)$ into $P$. It is noteworthy, that under the hypotheses of Theorem 1, we will be able to show not only the existence of a function $\varphi_0$ minimizing $\rho(\varphi, T)$, but also that $\varphi_0$ may be taken to be continuous.

## The Existence of a Best Approximation for $T$

The proof of Theorem 1 is modeled to some extent on a recent result of Olech [2] concerning best approximations to set-valued functions. His theorem was proved for functions mapping into a uniformly convex Banach space, but since the Banach space $C(X)$ is not uniformly convex, our proof departs considerably from his. We do retain the following notation of Olech:

$$r(f, T) = \inf_{p \in P} \sigma(p, Tf), \tag{5}$$

$$r(T) = \sup_{f \in C(X,R)} r(f, T). \tag{6}$$

Let $Y$ and $Z$ be topological spaces and let $F: Y \to 2^Z$. $F$ is called upper semicontinuous (u.s.c.) if $\{y \mid F(y) \subset G\}$ is open in $Y$ for each open set $G$ in $Z$ and lower semicontinuous (l.s.c.) if $\{y \mid F(y) \cap G \neq \varnothing\}$ is open in $Y$ for each open set $G$ in $Z$. Let $\varphi: Y \to Z$; we call $\varphi$ a selection for $F$ if $\varphi(y) \in F(y)$ for all $y$ in $Y$. If $Z$ is a linear topological space, we let $K(Z)$ denote the set of closed convex subsets of $Z$.

The following theorem of Michael [1] plays a key role in both the theorem of Olech and our Theorem 1.

THEOREM (Michael). *The following properties of $T_1$ spaces are equivalent:*

(a) *$Y$ is paracompact;*

(b) *If $E$ is a Banach space then every* l.s.c. *map $F$ of $Y$ into $K(E)$ has a continuous selection.*

THEOREM 1. *Suppose $Tf \neq \varnothing$ for each $f$ and suppose $T$ is* u.s.c. *Then there exists a continuous function $\varphi_0 : C(X, R) \to F$ that minimizes $\rho(\varphi, T)$ among all point functions $\varphi$ from $C(X, R)$ to $P$.*

*Proof.* If $f \in C(X, R)$ then $p \in Tf$ implies $\| p \| \leqslant 2R$. Therefore, $r(f, T) \leqslant 4R$ which insures us that $r(T)$ is finite. Let

$$\bar{B}(p, t) = \{q \in P \mid \| p - q \| \leqslant t\},$$

and for each $f$ in $C(X, R)$, let

$$H(f) = \{ p \in P \mid Tf \subseteq \bar{B}(p, r(T))\}. \tag{7}$$

We will show for each $f$ that $H(f)$ is closed, convex, nonempty and that $H$ is l.s.c.

Given any $\lambda > 0$,

$$\sigma(p, T\lambda f) = \sigma(p, \lambda Tf) = \sup_{y \in \lambda Tf} \| p - y \| = \sup_{z \in Tf} \| p - \lambda z \| = \lambda \sigma(\lambda^{-1} p, Tf).$$

Thus, $r(\lambda f, T) = \lambda r(f, T)$. If $f \in C(X, R)$, then there is $\lambda > 1$ such that $\lambda f \in C(X, R)$. This, together with $r(\lambda f, T) = \lambda r(f, T)$ means that $r(f, T) < r(T)$ for each $f$. Furthermore, if $\sigma(p, Tf) \leqslant r(f, T) + \epsilon$, $\epsilon > 0$, then

$$Tf \subseteq \bar{B}(p, r(f, T) + \epsilon).$$

Hence, $H(f)$ is nonempty for each $f$ in $C(X, R)$. It is straightforward to verify that $H(f)$ is closed and convex for each $f$.

To see $H$ is l.s.c., let $G$ be open in $P$ and let $p_0 \in H(f_0) \cap G$. As $r(f_0, T) < r(T)$ there is $\delta > 0$ and $q_0 \in H(f_0)$ such that $\| q_0 - y \| \leqslant r(T) - \delta$ for all $y \in Tf_0$. For $0 < \lambda < 1$, $\| \lambda q_0 + (1 - \lambda) p_0 - y \| \leqslant r(T) - \lambda\delta$. Let $h_0 = \lambda q_0 + (1 - \lambda) p_0$, then if $\lambda$ is small we have $h_0 \in G$ and moreover $h_0 \in H(f_0)$ by convexity. Let $B$ denote the open sphere centered at $h_0$ with radius $r(T)$. Since $\sigma(h_0, Tf_0) \leqslant r(T) - \lambda\delta$ then $Tf_0 \subseteq B$. Since $T$ is u.s.c. there is $\mu > 0$ such that $\| f - f_0 \| < \mu$ implies $Tf \subseteq B$. So $h_0 \in H(f)$ for all $f$ satisfying $\| f - f_0 \| < \mu$ and therefore $H$ is l.s.c.

Clearly, $P$ is a Banach space since it is a closed linear subspace of $C(X)$. Moreover, $C(X, R)$ is paracompact so we can apply Michael's theorem to assert the existence of a continuous function $\varphi_0$ such that $\varphi_0(f) \in H(f)$ for all $f \in C(X, R)$. Thus, by (7), $\rho(\varphi_0, T) \leqslant \rho(\varphi, T)$ for all point-valued functions $\varphi : C(X, R) \to P$.

The hypotheses of Theorem 1 can be met in many important cases. For example, if $P$ is finite dimensional then $P$ is approximatively compact. Thus, as a special case of a theorem of Singer [3], the operator $T$ is u.s.c.

## References

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